Topology of plasma equilibria and the current closure condition

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A virtually complete description of the topology of stationary incompressible Euler flows and the magnetic field satisfying the magnetostatic equation is given by a theorem due to Arnol'd. We apply this theorem to describe the topology of stationary states of plasmas with significant fluid flow, obeying the Hall magnetohydrodynamics model equations. In the context of the integrability (nonchaotic topology) of the magnetic and velocity fields, we discuss the validity of conditions analogous to that of Greene and Johnson, which, in the case of magnetostatic equations, states that the line integral $\oint dl/B$ is the same for each closed magnetic field line on a given magnetic surface. We also show how this property follows from the existence of a continuous volume-preserving symmetry of the magnetic field.

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I. INTRODUCTION

We apply certain ideas from differential geometry to the equilibrium state of a confined plasma. First we consider a conventional static equilibrium, characterized by force balance in the form

$$\mathbf{j} \times \mathbf{B} = \nabla p. \tag{1}$$

Here $\mathbf{B}$ is the magnetic field (its basic characteristic being $\nabla \cdot \mathbf{B} = 0$), $\mathbf{j} = (1/\mu_0)\nabla \times \mathbf{B}$ the plasma current, and p is the plasma pressure. In an effective magnetic confinement system, the surfaces of uniform pressure form a family of nested tori; these surfaces are called magnetic surfaces because the field lines lie on them. Surfaces on which the winding number of the field lines is rational are called rational surfaces and are especially fragile to various plasma instabilities.

Starting from the system (1) of magnetostatic equations [1], Greene and Johnson [2] showed that the rational surfaces possess the following property (see also [3]): For each closed magnetic field line L on a given rational surface the line integral

$$U = \oint_L \frac{dl}{B} \tag{2}$$

takes the same value and, therefore, is a surface quantity (i.e., depends only on the magnetic surface on which the particular field line lies). This property is called the *current closure condition*. As the argument makes essential use of (1), it seems inapplicable to more general plasma equilibria or to a vacuum magnetic field supported by external coils exclusively. Furthermore, as pointed out by Grad [4], the latter in general does not satisfy the current closure condition.

Using a theorem due to Arnol'd, we show that an argument very close to that of Greene and Johnson may be carried through without (1), providing that the magnetic field exhibits a spatial volume-preserving symmetry. In particular, the argument could be applied to an integrable (nonchaotic) vacuum (harmonic) field having a volume-preserving symmetry.

Further, we extend the discussion about integrability to a more general class of stationary states, in which plasma flow plays an essential role. Such a plasma is described by the Hall magnetohydrodynamics (MHD) model. The stationary states are not *a priori* integrable, even if the plasma pressure is not everywhere constant. We explore the conditions for the integrability of magnetic and velocity fields and the existence of surfaces invariant under the flow that are necessary for the confinement of plasma obeying these model equations. We also discuss the validity of the current closure and similar conditions for these fields.

The paper is organized as follows. In Sec. II, we recall Arnol'd's theorem concerning the integrability of divergence-free flows. In Sec. III, we show how the current closure condition (2) is related to the symmetry of the magnetic field. We discuss the effects of nonintegrability as the physically most relevant reason for violation of the current closure condition by stellarator magnetic fields [5–7]. In Sec. IV, we discuss the integrability of stationary states of the Hall MHD model.

II. ON THE INTEGRABILITY OF VOLUME-PRESERVING FLOWS

A general dynamical system of $2n$ ($n \in \mathbb{N}$) ordinary differential equations is integrable if one knows $2n-1$ independent integrals of motion. If a given dynamical system is Hamiltonian, it is often sufficient to know only n first integrals.

Liouville proved that if a Hamiltonian system with n degrees of freedom has n almost everywhere independent integrals of motion which are in involution, then the system is integrable by quadratures. In addition, he proved that if a level set of the integrals is a compact and connected manifold, then it is diffeomorphic to an n-dimensional torus. For some values of the integrals of motion, they cease to be independent and the level set ceases to be a manifold. Such critical values of the integrals correspond to separatrices dividing the phase space into the regions foliated by tori. In each of these regions, action-angle variables can be introduced.

For three-dimensional analytic divergence-free vector fields, Arnol'd proved the following fundamental result [8]:

Theorem 1. Let $\mathbf{V}$ and $\mathbf{W}$ be two analytic vector fields in a connected domain $D\subset\mathbb{R}^3$ bounded by a compact analytic surface to which $\mathbf{V}$ is parallel. If $\mathbf{V}$ and $\mathbf{W}$ are both divergence-free, commute with each other, i.e., $[\mathbf{V},\mathbf{W}]=\mathbf{0}$, and are not everywhere collinear in the given domain, i.e., $\mathbf{V}\times\mathbf{W}\not\equiv\mathbf{0}$, then the domain D is partitioned into a finite number of cells. Each of the cells is fibered either into tori or into annuli (cylinders), $I^1\times S^1$ (I^1 being an interval in $\mathbb{R}$). These surfaces are invariant under the flow of the vector fields $\mathbf{V}$ and $\mathbf{W}$. On an invariant torus, the field lines are either all closed or all dense. On an annulus, all trajectories are closed.

A version of this theorem, in which $\mathbf{W}=\nabla\times\mathbf{V}$, has been applied to hydrodynamics of ideal fluids [9,10]. It also applies to a magnetic field obeying the magnetostatic system of equations.

When considering the system of magnetostatic equations (1), $\mathbf{B}$ and $\mathbf{j}$ are implicitly assumed to be analytic vector fields and p an analytic function in a domain $D\subset\mathbb{R}^3$ bounded by an analytic surface. One can also assume that the magnetic field is tangential to the boundary surface. Such a situation obviously resembles the usual tokamak setup. Writing the commutator of two vector fields as

$$[\mathbf{V},\mathbf{W}]=\mathbf{V}\cdot\nabla\mathbf{W}-\mathbf{W}\cdot\nabla\mathbf{V}, \tag{3}$$

and using the classical vector identity

$$\nabla\times(\mathbf{V}\times\mathbf{W})=\mathbf{V}\nabla\cdot\mathbf{W}-\mathbf{W}\nabla\cdot\mathbf{V}-\mathbf{V}\cdot\nabla\mathbf{W}+\mathbf{W}\cdot\nabla\mathbf{V}, \tag{4}$$

we can see that the vector fields $\mathbf{B}$ and $\mathbf{j}$, satisfying the magnetostatic equations (1), commute, i.e.,

$$[\mathbf{B},\mathbf{j}]=-\nabla\times(\mathbf{B}\times\mathbf{j})=\mathbf{0}. \tag{5}$$

The plasma pressure p is the first integral for the vector fields $\mathbf{B}$ and $\mathbf{j}$ whose field lines are constrained to the level sets of p. If the magnetic field $\mathbf{B}$ and the current density $\mathbf{j}$ are not everywhere collinear in the given domain, the topology of these level sets is determined by Arnol'd's theorem. The domain D is partitioned into a finite number of cells, foliated by tori or annuli that are the level sets of the pressure function.

The existence of nested pressure surfaces that constrain the magnetic field lines, embedded in the system of magnetostatic equations, is necessary for the confinement of the charged particles, which in the lowest approximation follow the magnetic field lines.

Generic three-dimensional nonintegrable equilibria satisfying the magnetostatic equations (1) must violate some of the assumptions of Arnol'd's theorem. One possibility is violation of the analyticity requirement for the vector fields $\mathbf{B}$ or $\mathbf{j}$, and thus the analyticity of p that implies the existence of only a finite number of critical points ($\nabla p=\mathbf{0}$) and the existence of the finite number of cells. The other possibility is that the equilibrium is force-free, i.e., $\mathbf{j}=\lambda\mathbf{B}$ everywhere in the given domain. If λ is constant in the given domain, then the fields can have more complicated topology. In that case the field is an eigenfunction of the curl operator (Beltrami field).

III. THE CURRENT CLOSURE CONDITION

Starting from the system of magnetostatic equations (1) and the assumption that the domain $D\subset\mathbb{R}^3$ is foliated by nested toroidal pressure surfaces, Greene and Johnson [2] showed that the integral (2) is the same for each closed magnetic field line on the same surface. This property is also implied by the existence of a volume preserving symmetry of the magnetic field. If an analytic magnetic field $\mathbf{B}$ has a spatial volume-preserving symmetry group generated by an analytic infinitesimal generator $\mathbf{W}$ and satisfies the assumptions of Arnol'd's theorem, then inside each of the cells whose existence is guaranteed by the same theorem, the current closure condition is valid. If a magnetic field $\mathbf{B}$ admits a spatial volume-preserving symmetry group with an infinitesimal generator $\mathbf{W}$, then there exists a function α, such that

$$\mathbf{W}\times\mathbf{B}=\nabla\alpha. \tag{6}$$

Following [2], we can then calculate the flux of the generator $\mathbf{W}$ through an elementary surface containing a magnetic field line and bounded by two infinitesimally separated toroidal (or cylindrical) magnetic surfaces inside a single cell,

$$\delta\Phi=\int_{\delta S}\frac{\mathbf{B}\times\nabla\alpha}{B^2}d\mathbf{S}, \tag{7}$$

and, thus

$$\delta\Phi=\frac{d\alpha}{dV}\delta V\oint_L\frac{dl}{B}. \tag{8}$$

Here, V is the volume enclosed by the toroidal magnetic surface (or the volume enclosed by the cylindrical magnetic surface and the boundary of D) and, as earlier, B is the magnitude of $\mathbf{B}$. As the generator $\mathbf{W}$ is divergence-free and the rotation number of each magnetic field line on the given magnetic surface is the same, the total flux of $\mathbf{W}$ flowing across each of them must be the same. This shows that the line integral (2) is line independent on the given magnetic surface. Though simple, this observation is important when treating more general classes of plasma or fluid equilibria.

One can explicitly show that magnetic fields exhibiting translational, axial, and helical symmetry are integrable:

Example 1. If $\mathbf{B}$ does not depend on the z coordinate, take $\mathbf{W}=\hat{\mathbf{z}}$. It is easy to show that $[\mathbf{B},\mathbf{W}]=\mathbf{0}$, and that all "two-dimensional" vector fields are integrable.

Example 2. For the vector field $\mathbf{B}$, which does not depend on the angle ϕ in cylindrical coordinates, $\mathbf{W}=r\hat{\boldsymbol{\phi}}$. Again, $[\mathbf{B},\mathbf{W}]=\mathbf{0}$. This is the well-known result of the integrability of vector fields that do not depend on the azimuthal angle ϕ.

Example 3. The generator of helical symmetry is $\mathbf{W}=\omega\hat{\mathbf{z}}+r\hat{\boldsymbol{\phi}}$ (ω being a constant). We easily find that $[\mathbf{B},\mathbf{W}]=\mathbf{0}$ for the vector fields depending only on r and the helical angle $\theta=z-\omega\phi$. Notice that as $\mathbf{B}$ is periodic in ϕ with period 2π, it is periodic in θ with period $2\pi\omega$. One can now identify two different $\omega z+\phi=\text{const}$ surfaces, and the surfaces $\theta=0$ and $\theta=2\pi\omega$, to make a toroidal domain.

As the above symmetries are volume preserving, these magnetic fields obey equations similar to (1) and consequently the condition (2) is valid.

Experimental violation of the current closure condition is related to the breaking of the integrability and thus a volume-preserving symmetry of the magnetic field. Integrable fields are very special. A typical perturbation of an integrable system destroys the global integrability of the magnetic field. In the cells foliated by tori the scenario resembles the transition to chaos in Hamiltonian flows. Though some tori break for arbitrarily small perturbations, a remarkable theorem, due to Kolmogorov, Arnol'd, and Moser, guarantees the preservation of smooth Diophantine tori in systems sufficiently close to an integrable one. The remaining tori still form a set of positive measures for sufficiently small perturbations. Generically, there are no rational surfaces. In place of the rational surfaces, island chains appear (most clearly at low order rationals). Though there are still some periodic orbits (of the same winding number) remaining after the breakup of a rational surface, they no longer lie on the same magnetic surface. Consequently, one should not expect that the integral (2) taken along them should still have the same value [6].

Violation of the current closure condition (2) necessarily means a violation of the magnetohydrostatic equation (1) with nonconstant pressure function. The latter enforces the integrability of the magnetic field, forcing current closure to hold.

IV. TOPOLOGY OF STATIONARY STATES OF PLASMA WITH FLOW

The system of equations (1) doesn't take into account ion flow and thus may not be appropriate to describe situations with a significant plasma flow. In this section, we analyze the stationary states of a two-component infinitely conducting plasma, consisting of electron and ion fluids.

The stationary states of such a two-component plasma are described by the following system of equations:

$$\begin{aligned}(\mathbf{B}+\nabla\times\mathbf{v})\times\mathbf{v}&=-\frac{\nabla p_i}{\rho}-\nabla\left(\frac{v^2}{2}+\varphi\right),\\ \left(\mathbf{v}-\frac{\mathbf{j}}{\rho}\right)\times\mathbf{B}&=-\frac{\nabla p_e}{\rho}+\nabla\varphi,\\ \nabla\times\mathbf{B}&=\mu_0\mathbf{j},\\ \nabla\cdot\mathbf{B}&=0,\\ \nabla\cdot(\rho\mathbf{v})&=0.\end{aligned}\tag{9}$$

Here we use the standard Alfvén notation and normalization of Mahajan and Yoshida [11]: ρ is the dimensionless number density; p_i and p_e are, respectively, the ion and electron betas; $\mathbf{v}$ is the ion fluid velocity normalized to the Alfvén speed; the electron speed is determined via $\mathbf{j}=(1/\mu_0)\nabla\times\mathbf{B}=\rho(\mathbf{v}-\mathbf{v}_e)$; and φ is the electrostatic potential. These equations determine the stationary states of a model of quasineutral plasma dynamics obtained from the equations of motion of the ion and the electron fluids, the continuity equations of each species and Maxwell's equations, in the limit $m_e\to 0$. In order to obtain a closed system of equations, one can for example take $p_i=p_e$ and $\nabla\varphi=\mathbf{0}$ in (9).

In light of the preceding discussion, we now delineate the sufficient conditions for integrability (nonchaoticity) of the magnetic and/or velocity fields. We assume that all fields involved in (9) are analytic on a domain $D\subset\mathbb{R}^3$ bounded by a compact analytic surface. The existence of solutions to the related boundary value problems is a different question that we do not address here. We only address the question of the topological properties of the stationary states which are solutions of the two-fluid model (9), in the case of incompressible and compressible plasmas.

First, we observe that in the case of an incompressible plasma ($\rho=\text{const}$): (i) the right-hand sides of the first two equations become total gradients of the functions $\psi=-p_i/\rho-v^2/2-\varphi$ and $\chi=-p_e/\rho+\varphi$, respectively; (ii) all vectors involved in the cross product are divergence-free.

If we assume that ψ and χ are not everywhere constant and that $\mathbf{v}$ and $\mathbf{B}$ are tangential to the boundary of the space occupied by the plasma, we can easily see that both the magnetic and the velocity fields are integrable.

In fact, even in the general case of a compressible plasma, the model (9) allows for only a nonchaotic magnetic field, provided that the electron pressure p_e is not everywhere constant and that $\nabla\varphi$ is zero or a function of the density ρ only. In that case, the model implies that $\rho\mathbf{v}_e\times\mathbf{B}=-\nabla p_e$, with the divergence-free vector fields $\rho\mathbf{v}_e$ and $\mathbf{B}$. Arnol'd's theorem then gives a virtually complete description of the topology of the analytic magnetic and the electron velocity fields. The domain $D\subset\mathbb{R}^3$ is partitioned into a finite number of cells. Inside each of the cells, the lines of the vector fields $\mathbf{B}$ and $\mathbf{v}_e$ are constrained to two-dimensional toroidal or cylindrical surfaces that are the noncritical level sets of the electron pressure p_e.

In the two-fluid model, the closure condition (2) is satisfied by the integrable $\mathbf{B}$ field, although the commuting vector field is not the current density, but the product of the density ρ and the electron velocity field $\mathbf{v}_e$. The electron current density field satisfies similar closure condition. The line integral

$$\oint_{L_{\mathbf{v}_e}}\frac{dl}{\rho v_e}\tag{10}$$

is independent of the closed electron velocity field line $L_{\mathbf{v}_e}$ on a given magnetic surface. In the case of homogenous plasma, the closure conditions analogous to (2) are satisfied by the integrable velocity fields.

A sufficient condition for integrability of the ion velocity field $\mathbf{v}$, and thus the current density $\mathbf{j}$, is that v^2 is a function of the density ρ only and that the ion pressure is not everywhere constant (assuming that $\nabla\varphi$ is zero or function of the density ρ only, as before). This can easily be seen if the first of equations (9) is multiplied by ρ. In that case, the ion flow is constrained to ion isobaric surfaces (level sets of p_i), which are almost all topological tori or cylinders.

A particular case of the above situation is the one in which the plasma pressures are functions of the density only. In that case, the right-hand sides of the first two equations in (9) can be written as total gradients of some analytic functions, which we will assume are not constant. The integrability of the magnetic and the velocity fields, necessary for plasma confinement, is then guaranteed under the following condition:

$$\nabla \rho \cdot \mathbf{v} = 0. \tag{11}$$

This condition states the ion flow lies on the isopycnic surfaces. Since in that case the isopycnic surfaces are also isobaric and, from the second of equations (9), it follows that $\nabla p_e \cdot \mathbf{v}_e = 0$, we also have the condition $\nabla \rho \cdot \mathbf{j} = 0$. The condition that the current density $\mathbf{j}$ lies on isobaric surfaces also is the same as the one coming from the system of magnetostatic equations (1). Additionally, the isopycnic surfaces are the surfaces of constant electrostatic potential also, and the allowed ion velocity field must satisfy $v^2 = v^2(\rho)$.

Let us now consider an example of such an equilibrium that satisfies the system (9), with a simplified cylindrical geometry, as is usual in plasma physics considerations. Consider the magnetic field $\mathbf{B} = B_\phi(r)\hat{\boldsymbol{\phi}} + B_z(r)\hat{\mathbf{z}}$, depending only on the cylindrical coordinate r, in the region $D \subset \mathbb{R}^3$ bounded by two coaxial cylindrical surfaces $r = R_1 > 0$ and $r = R_2 > 0$. The field is independent of z, and thus satisfies the periodic boundary conditions at $z = 0$ and $z = z_0$. Consequently, a field line can be either closed or dense, with a rational or irrational winding number $\omega = B_z(r)/B_\phi(r)$, $B_\phi(r) \neq 0$ $(R_1 \leqslant r \leqslant R_2)$, respectively (a toroidal cell). If one takes $B_z(r) = 0$, and considers the magnetic field, which is thus additionally tangential to the planes $z = 0$ and $z = z_0$, all the magnetic field lines are closed (a cylindrical cell). The field is divergence-free and has the property that $\nabla \times \mathbf{B} = -\partial_r B_z(r)\hat{\boldsymbol{\phi}} + (1/r)\partial_r[rB_\phi(r)]\hat{\mathbf{z}}$ has the same form as $\mathbf{B}$. A self-consistent solution of the system (9) can be constructed using the ion and electron velocity fields of the form $\mathbf{v} = v_\phi(r)\hat{\boldsymbol{\phi}} + v_z(r)\hat{\mathbf{z}}$ and $\mathbf{v}_e = \mathbf{v} - (1/\mu_0\rho)\nabla \times \mathbf{B}$. The plasma pressures p_i and p_e, the density ρ, and the electrostatic potential φ are functions of the coordinate r only. The ion and electron velocity field lines and the magnetic field lines lie on isopycnic surfaces.

In particular, one can consider a "vacuum" magnetic field $\mathbf{B} = (1/r)\hat{\boldsymbol{\phi}} + B_0\hat{\mathbf{z}}$, with B_0 a constant. Such a field has $\nabla \times \mathbf{B} = \mathbf{0}$, but the electron and ion velocity field lines lie on cylindrical surfaces (with the periodic boundary conditions).

The above examples of the integrable fields, however, involve high symmetrical fields that do not depend on the cylindrical coordinates ϕ and z. These situations are not generic. One possibility for the violation of the condition for integrability of the involved fields are generically chaotic double-Beltrami flows [11]. They satisfy the system of equations (9) with the gradients on the right-hand side of the first two of the equations equal to zero identically.

At the end of this section, let us mention that the first and the last of the equations (9), in the case $\mathbf{B} = \mathbf{0}$, correspond to the well-known equations of fluid mechanics describing the stationary states of Euler fluids. The discussion about the current closure condition naturally extends to this case, leading to a *vorticity closure condition* for stationary incompressible Euler flow described by the equation $\mathbf{w} \times \mathbf{v} = -\nabla p/\rho - \nabla v^2/2$, where $\mathbf{w} = \nabla \times \mathbf{v}$ is the vorticity.

V. SUMMARY

By invoking Arnol'd's theorem, we have shown how the current closure condition follows from a spatial continuous volume-preserving symmetry of the magnetic field. By identifying the conditions under which the magnetic and velocity fields are integrable in plasmas with flows, we have established the minimum ground rules for confinement studies in systems beyond those described by ordinary magnetohydrodynamics.

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